Dynamics of Thin Liquid Films in the Presence of Surface Tension Gradients

JOHN E. BUCHANAN

University of New South Wales
Kensington, N. S. W. 2033, Australia

In a recent article, Ludviksson and Lightfoot (1971) outlined many of the properties of the creeping laminar flow of a liquid film climbing a vertical surface under the influence of a surface tension gradient. The terms of their description, however, are unnecessarily complex and they obscure the satisfying simplicity of some of the relations.

Thus, for instance, the mean fluid velocity at any cross-section of the film is given by their equation (7),

$$N_{Re} = \frac{1}{2} R^2 N_{Th} - \frac{1}{3} R^3 \qquad (1)$$

and a graph is presented showing the family of curves of N_{Re} plotted against the group R with N_{Th} as parameter.

Now a simple dimensional analysis applied to the variables used confirms that three dimensionless variables are required. But the appearance of a Reynolds number is anomalous. It can never be a variable in a case, like this, of unaccelerated laminar flow, where inertial mass is irrelevant. Closer examination reveals that only gravitational mass is relevant here, and that the product ρg, the specific weight, should be treated as a single variable. It appears as such in the differential equations presented.

If the dimensional analysis is now applied and velocity is confined to the dependent variable, the only possible independent variable is $\rho gh/\gamma$, which I shall call B. Viscosity cannot be accommodated in the independent variable; it serves only to scale forces and velocities. Equation (1) now takes on the simple linear form

$$V = 1/2 - B/3 \qquad (2)$$

To describe the point velocity, a further geometric variable is needed to define distance from the solid surface. Then the velocity profile is

$$U = Y\,(1 - B) + B\,Y^2/2 \qquad (3)$$

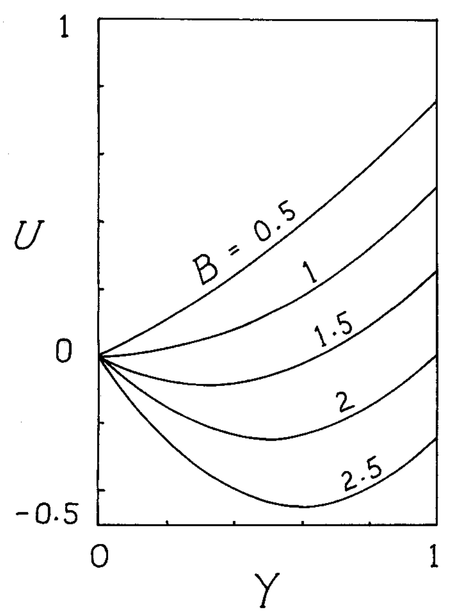

Fig. 1. Velocity profiles.

These parabolic profiles are shown in Figure 1. They are, *mutatis mutandis,* the familiar profiles occurring in plane Couette flow. At the liquid surface, the velocity is given by

$$U_h = 1 - B/2 \qquad (4)$$

and the dimensionless velocity gradient dU/dY is equal to unity.

In the usual way, the group B may be interpreted as a simple force ratio. It is the ratio of the weight force per unit area of the film to the surface tension force per unit area. In terms of the groups used in the original paper

$$B = R\,N_{Th} \qquad (5)$$

The maximum volumetric flow for a fixed value of the surface tension gradient is also of interest. Clearly, $V_{\max}$ is a function of B. But $q_{\max}$ cannot depend upon h. Rather, h is a connected dependent variable. Therefore, necessarily, $V_{\max} \cdot B^2$ is a constant. Putting $Q = V\,B^2$, it may easily be verified that $Q_{\max} = 1/6$; it occurs for a film thickness such that $B = 1$.

For the calculation of rates of motion of the film, the general relation between this dimensionless volumetric flow rate Q and the dimensionless film thickness is also useful. It is

$$Q = B^2/2 - B^3/3 \qquad (6)$$

The flow is zero for $B = 3/2$. For values of B up to that level, net flow is upwards, for higher values liquid flows down.

Using the same assumptions as the original authors, the upwards velocity of the level at which a certain film thickness appears may be shown to be dq/dh or, again in dimensionless terms,

$$W = (\rho g \mu/\gamma^2) \cdot dq/dh = dQ/dB \qquad (7)$$

and

$$W = B - B^2 \qquad (8)$$

The velocity W has a maximum value of 1/4 at $B = 1/2$, the point of inflection of the $Q - B$ curve.

Because a Marangoni number had already been defined by several previous workers, Ludviksson and Lightfoot amended an earlier nomination and called their group $(\gamma^3/\rho g^2 \mu^2)^{1/3}$ the Thomson number, for J. J. Thomson (sic) who first correctly explained the behavior of wine tears and some other similar phenomena. (In fact, it was not Joseph John Thomson (1856-1940) but James Thomson (1822-92), eldest brother of William Thomson, Lord Kelvin.) But, according to the compilation by Catchpole and Fulford (1966), this name has also been claimed, though it is not clear which Thomson was so honored. Considering its form and its utility, perhaps the group should be left anonymous.

NOTATION

B = $\rho gh/\gamma$, dimensionless film thickness
g = gravitational acceleration, m/s^2
h = film thickness, m
N_{Re} = $h <v> \rho/\mu$, film Reynolds number
N_{Th} = $(\gamma^3/\rho g^2\mu^2)^{1/3}$, Thomson number
q = volumetric flow/unit width, m^2/s
Q = $q\, \rho^2 g^2 \mu/\gamma^3$, dimensionless flow rate
R = $h(g\rho^2/\mu^2)^{1/3}$
U = $v\, \mu/\gamma h$, dimensionless fluid velocity
v = fluid velocity, m/s
$<v>$ = q/h = mean fluid velocity at a cross-section, m/s
v_h = velocity of a thickness locus, m/s
V = $<v>\mu/\gamma h = q\mu/\gamma h^2$ = dimensionless mean velocity
W = $v_h\, \rho g \mu/\gamma^2$, dimensionless locus velocity
y = distance from solid surface, m
Y = y/h = dimensionless distance from surface

Greek Letters

γ = surface tension gradient, N/m^2
μ = dynamic viscosity, Pa s
ρ = density, Kg/m^3

LITERATURE CITED

Catchpole, J. P., and G. Fulford, "Dimensionless Groups," *Ind. Eng. Chem.*, **58**, (3), 46 (1966).

Ludviksson, V., and E. N. Lightfoot, "The Dynamics of Thin Liquid Films in the Presence of Surface-Tension Gradients," *AIChE J.*, **17**, 1166 (1971).

Manuscript received July 27, 1972; note accepted September 15, 1972.

A Simple Expression for the Velocity Distribution in Turbulent Flow in Smooth Pipes

STUART W. CHURCHILL and BYUNG CHOI

School of Chemical Engineering
University of Pennsylvania, Philadelphia, Pennsylvania 19104

The velocity distribution in turbulent flow in smooth pipes is of great interest because it provides a test of the attainment of fully developed flow, necessary information for the detailed calculation of heat transfer, component transfer, and chemical conversions in turbulent flow, and the basis for the development of the highly successful analogies between momentum transfer and heat and component transfer.

For the region adjacent to the wall the assumption of laminar motion and a negligible variation in shear stress leads to

$$u^+ = y^+ \tag{1}$$

Prandtl (1933) derived the expression

$$u^+ = A + B \ln y^+ \tag{2}$$

by assuming that the turbulent shear stress was proportional to $(ydu/dy)^2$ and that the viscous shear stress and the radial variation in the total shear stress were negligible. Despite these extreme assumptions, Equation (2) represents the form of the velocity distribution far outside the laminar boundary reasonably well.

Equations (1) and (2) can be patched together to yield a complete representation by the proper choice of A and B. A considerably better representation is obtained by using Equation (1) out to $y^+ = 5$, Equation (2) with one set of coefficients from 5 to 30 and with a second set

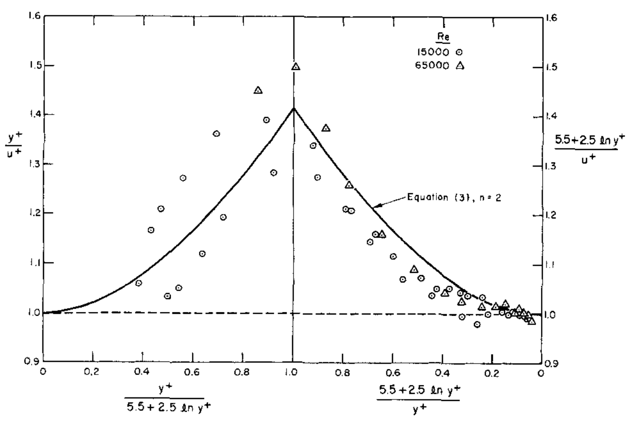

Fig. 1. Graphical form for development of correlation.

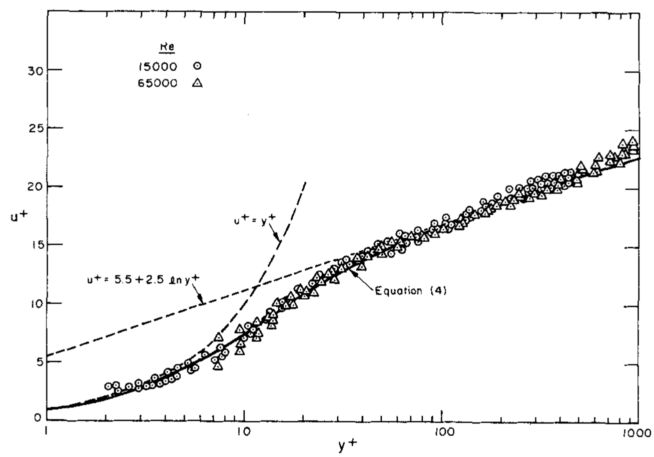

Fig. 2. Demonstration of correlation in conventional graphical form.